$\boxed{\text{Accuracy}}$

1) $x(t+\Delta t) = x(t) + \Delta t\, f(x(t))$

2) $\dfrac{dx}{dt} = f(x)$

$$\frac{d^2x}{dt^2} = \frac{df}{dx}\frac{dx}{dt} = \frac{df}{dx} f$$

$$x(t+\Delta t) = x(t) + f(x)\Delta t + \frac{1}{2} f'(x) f(x) \Delta t^2 + \ldots$$

$$f\left(x + \frac{1}{2} f \Delta t\right) = f(x) + \frac{1}{2} f'(x) f(x) \Delta t$$

so $\quad x(t+\Delta t) = x(t) + k_2$

$$k_1 = f(x)\Delta t$$

$$k_2 = f\left(x + \frac{1}{2}k_1\right)\Delta t$$

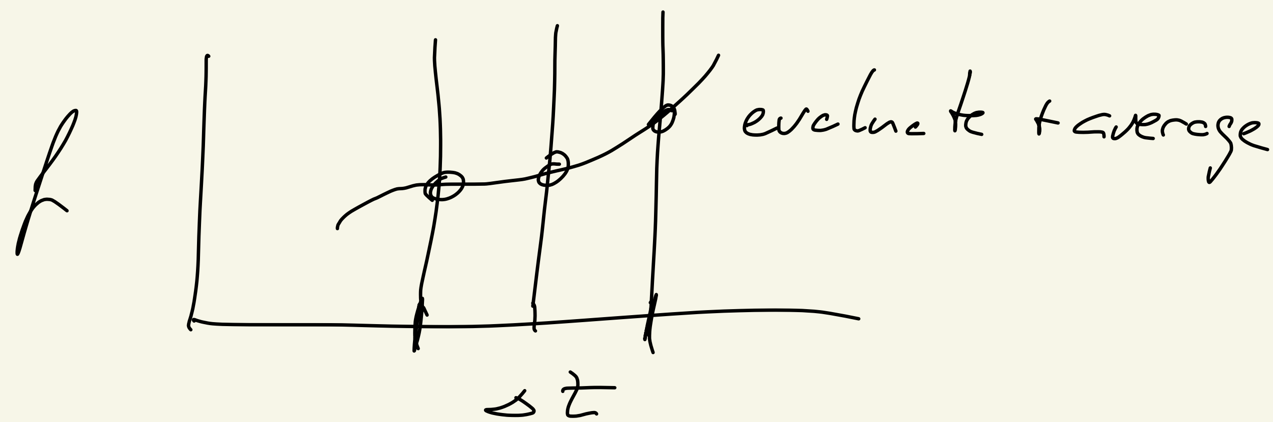
3)
$$x(t+\Delta t) = x(t) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = \Delta t\, f(x)$$

$$k_2 = \Delta t\, f\left(x + \tfrac{1}{2}k_1\right)$$

$$k_3 = \Delta t\, f\left(x + \tfrac{1}{2}k_2\right)$$

$$k_4 = \Delta t\, f\left(x + k_3\right)$$



4)
$$\tau \frac{dx}{dt} = a - x$$

$$x(t+\Delta t) = a + \left(x(t) - a\right)e^{-\Delta t/\tau}$$

$$\simeq a + \left(x(t) - a\right)\left(1 - \frac{\Delta t}{\tau} + \frac{1}{2}\frac{\Delta t^2}{\tau^2}\right)$$

$$= x(t) + (a - x(t))\frac{\Delta t}{\tau}$$

$$- \frac{1}{2}(a - x(t))\frac{\Delta t^2}{\tau^2}$$

$$f = \frac{a - x}{\tau} \qquad f' = -1/\tau$$

so $\quad x(t) + f(x)\Delta t + \frac{1}{2} f(x) f'(x) \Delta t^2$

5) $\boxed{\text{Stability}}$

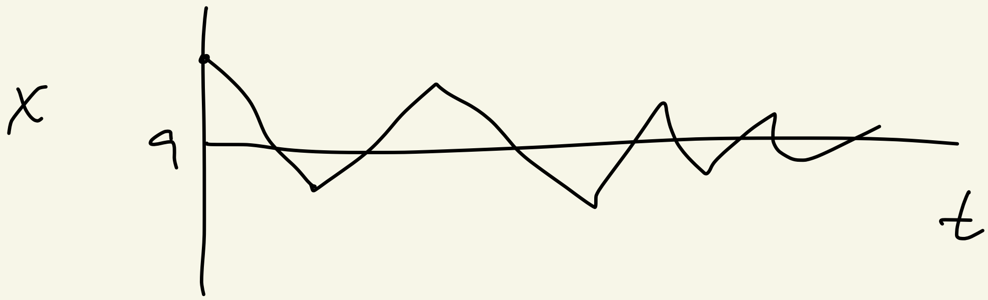
$$\tau \frac{dx}{dt} = -x \qquad \frac{dx}{dt} = -\frac{x}{\tau}$$

$$x(t + \Delta t) = x(t)\left(1 - \frac{\Delta t}{\tau}\right)$$

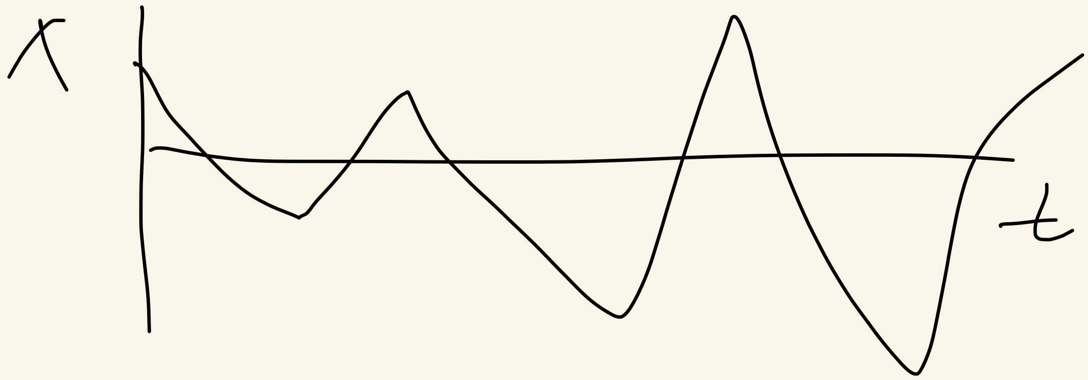
What if $\quad \dfrac{\Delta t}{\tau} > 1$

$$x(t + \Delta t) \longrightarrow 0$$

but instead



or even worse



How to prevent

a) $x(t+\Delta t) = x(t)e^{-\Delta t/\tau}$

$\frac{\Delta t}{\tau} \to \infty \qquad x(t+\Delta t) \to 0$ ✓

b) $x(t+\Delta t) = x(t) - x(t)\Delta t$ Euler

$\to x(t+\Delta t) = x(t) - x(t+\Delta t)\Delta t/\tau$

Reverse Euler

so $x(t+\Delta t)\left(1+\frac{\Delta t}{\tau}\right) = x(t)$

$$x(t+\Delta t) = \frac{x(t)}{1+\Delta t/\tau}$$

$$\Delta t/\tau \to \infty \qquad x(t+\Delta t) \to 0$$

but
$$\frac{1}{1+\frac{\Delta t}{\tau}} = 1 - \frac{\Delta t}{\tau} + \cdots$$

so
$$x(t+\Delta t) = x(t)\left(1-\Delta t/\tau\right)$$

first-order accurate

General method is like 2nd order R-K

$$k_1 = f(x)\Delta t$$

$$x(t+\Delta t) = x(t) + f(x+k_1)\Delta t$$

not $\frac{k_1}{2}$!

Important for Hodgkin-Huxley

$$h_1 = \Delta x$$

$$\Delta x = f(x + \Delta x)\Delta t$$

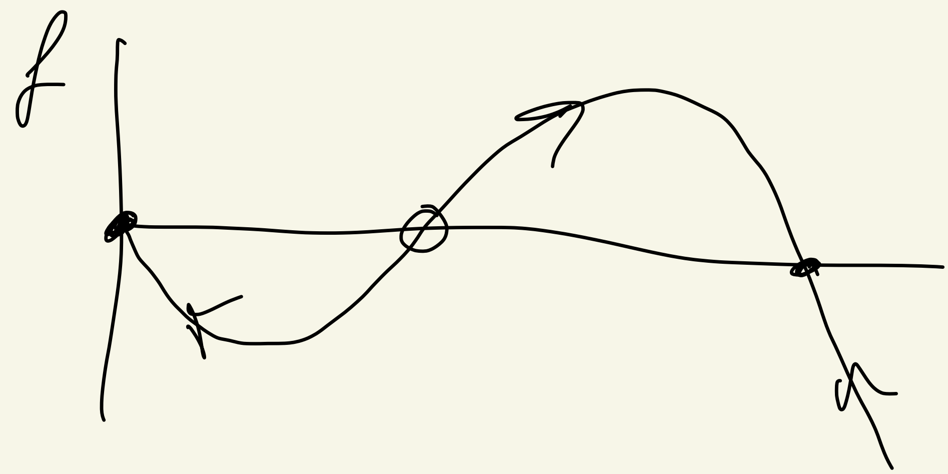
$$= f(x)\Delta t + f'(x)\Delta x \Delta t$$

$$\Delta x\left(1 - f'(x)\Delta t\right) = f(x)\Delta t$$

$$\Delta x = \frac{f(x)\Delta t}{1 - f'(x)\Delta t}$$

Linearization & Stability

$$\frac{dx}{dt} = f(x) \qquad f(x_0) = 0$$



$$x = x_0 + y$$

$$f(x) = f(x_0) + f'(x_0)\, y$$

$$\frac{dy}{dt} = f'(x_0)\, y$$

$$y(t) = y(0)\, e^{f'(x_0)\, t}$$